

Exact Bounded Boundary Zero-Controllability for the Two-Dimensional Wave Equation

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Abstract

The problem of the exact bounded control of oscillations of the two-dimensional membrane is considered. Control force is applied to the boundary of the membrane, which is located in a domain on a plane. The goal of the control is to drive the system to rest in a finite time.

Keyword: Controllability to rest, wave equation, boundary control, bounded control

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1 Introduction

The problem of exact boundary controllability of oscillations of a plane membrane is considered. Control force has the restriction on its absolute value. We will prove that the plane membrane can be driven to rest in a finite time. Exact mathematical definitions will be provided. It should be noted, that the given method for the proof in this article can be used in case of any other dimension, but here the two-dimensional case is provided for clear and simple presentation.

The problem of full stabilization in a finite time in case of the distributed control is described in the monograph [1]. This reference also contains the upper estimate for an optimal control time.

Earlier the question of the control of oscillations of a plane membrane by means of boundary forces is considered by many authors (i. g. overviews of D. L. Russell [2] and J. Lions [3], as well as the literature provided there). The monograph [4] describes the task of stabilizing oscillations of a restricted string by means of the boundary control, and proves that vibrations of the string can be driven to rest in a finite time under the condition of restriction imposed on an absolute value of the control function, and estimate is provided for the time that is necessary for full stop. In monograph [5] problems of the optimal control of systems with distributed parameters are studied and optimal conditions are stated, which are similar to conditions for systems with finite number of freedom's degrees. Although this method does not provide constructive technique for finding an optimal control in many cases. In synoptic article [3] the problem of exact zero-controllability of a membrane is considered, the existence of the boundary control is proved and the time estimate is given, which is required for driving to rest. Here authors while studying the problem in various formulations often reject the requirement of optimality of the control and solve only the problem of controllability, which is much easier. What is more, problems with restrictions of

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the force's absolute value are not considered, and explicit forms for control functions are not found, only theorems of existence are proved.

The statement of the problem in the article essentially differs from the one in [2] and [3], because the value of control force on the boundary has to satisfy the condition: $|u(t, x)| \leq \varepsilon$. Note, here the aim is to find not an optimal control, but the admissible (satisfying initial restrictions) control.

2 The statement of the problem

Let us consider the initial-boundary value problem for the two-dimensional wave equation:

$$w_{tt}(t, x) - \Delta w(t, x) = 0, \quad (t, x) \in Q_T = (0, T) \times \Omega, \quad (1)$$

$$w|_{t=0} = \varphi(x), \quad w_t|_{t=0} = \psi(x), \quad x \in \Omega, \quad (2)$$

$$\frac{\partial w}{\partial \nu} = u(t, x), \quad (t, x) \in \Sigma, \quad (3)$$

where Ω is a domain with a smooth boundary (including additional conditions, which are necessary for existence and uniqueness of the solution), ν — the outer normal to the boundary of the domain Ω , Σ is a lateral surface of a cylinder Q_T . Initial data $\varphi(x)$ and $\psi(x)$ are given and will be chosen in suitable Hilbert spaces, $u(t, x)$ is a control function defined on the boundary of Ω .

Let $\varepsilon > 0$ be an given arbitrary number. Let us impose the constraint on the control function:

$$|u(t, x)| \leq \varepsilon, \quad (4)$$

The problem is to construct a control $u(t, x)$ satisfying inequality (4) such that the corresponding solution $w(t, x)$ to the initial-boundary value problem (1)—(3) and its derivative with respect to t becomes zero at some time T , i.e.

$$w(T, x) = 0, \quad w_t(T, x) = 0, \quad (5)$$

for all $x \in \Omega$. If we obtained a control $u(t, x)$ such that conditions (5) are achieved then the system (1)—(3) is called *controllable to rest*.

The following theorem is the main result of this article.

Theorem 1. Let $\varphi(x) \in H^7(\Omega)$ and $\psi(x) \in H^6(\Omega)$ such that

$$\frac{\partial \varphi(x)}{\partial \nu} = 0, \quad \psi(x) = 0, \quad x \in \partial\Omega. \quad (6)$$

Then there are time T and the control function $u(t, x) \in C(\Sigma)$, satisfying the restriction (4), such that the system (1)—(3) is controllable to rest.

The proof of the theorem 1 consists of two steps. The first step stabilizes the considered solution and its first derivative with respect to t in a small vicinity of zero in the norm of $C^5(\overline{\Omega}) \times C^4(\overline{\Omega})$, and the second step allows to drive to rest the system in this small vicinity.

3 The first step of the control

Here we state the task to stabilize a pair $(w(t, x), w_t(t, x))$ in arbitrarily small vicinity of zero in the norm of the space $C^{5,4}(\overline{\Omega}) = C^5(\overline{\Omega}) \times C^4(\overline{\Omega})$ where $w(t, x)$ is a the solution to the system (1)—(3) and

$w_t(t, x)$ is its first derivative with respect to t . What is more the control function should satisfy the restriction (4).

We use the results of [2] for this purpose. The research considers a friction which is introduced on the boundary (or on a part of it) of Ω . The friction is given by the first derivative of $w(t, x)$ with respect to t , i.e. the initial-boundary value problem (1), (2) with boundary condition

$$\frac{\partial w(t, x)}{\partial \nu} = -k \frac{\partial w(t, x)}{\partial t}, \quad (7)$$

where $k > 0$ is a friction coefficient.

According to the theorem's condition we have $\varphi(x) \in H^7(\Omega)$ and $\psi(x) \in H^6(\Omega)$. It is known that in this case the solution $w(t, x)$ to the problem (1), (2), (7) such that $w(t, x) \in C([0, +\infty); H^7(\Omega))$ and $w_t(t, x) \in C([0, +\infty); H^6(\Omega))$. We note that it is necessary for $\varphi(x)$, $\psi(x)$ to satisfy the following condition:

$$\frac{\partial \varphi(x)}{\partial \nu} = -k \psi(x), \quad x \in \partial \Omega. \quad (8)$$

By virtue of (6) condition (8) is done **for any** k .

Let us consider the energy of the system (1), (2), (7):

$$E(t) = \int_{\Omega} (w_{x_1}^2(t, x) + w_{x_2}^2(t, x) + w_t^2(t, x)) dx.$$

In [2] is proven, because of the presence of friction on the boundary the dissipation of the system's energy occurs, that

$$\frac{dE(t)}{dt} \leq 0, \quad t \geq 0. \quad (9)$$

What is more the estimate is correct

$$E(t) \leq \frac{C(\varphi, \psi)}{1+t}, \quad t \geq 0, \quad (10)$$

where $C(\varphi, \psi)$ is a constant, depending on initial data of the system. Differentiating the equation (1), initial conditions (2), and also boundary condition (7) with respect to x_1 and x_2 , we obtain the estimate for $w(t, x)$:

$$\int_{\Omega} \sum_{0 < |\alpha| \leq 7} |D_x^\alpha w(t, x)|^2 dx \leq \frac{C_1(\varphi, \psi)}{1+t}, \quad t \geq 0, \quad (11)$$

$$\alpha = (\alpha_1, \alpha_2), \quad |\alpha| = \alpha_1 + \alpha_2, \quad D_x^\alpha = D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2}, \quad D_{x_j}^{\alpha_j} = \frac{\partial^{\alpha_j}}{\partial x_j^{\alpha_j}}.$$

Left side of the estimate (11) is truncated square of the norm in $H^7(\Omega)$ of the function $w(t, \cdot)$. In order to make the truncated norm a full one in $H^7(\Omega)$ we should get homogenous boundary conditions for the function $w(t, \cdot)$ itself and its derivatives with the respect to x_1 , x_2 including derivatives of the fifth order (Friedrichs' inequality). The operator of extension E is used for this purpose (see [6]). Let Ω_δ be δ -vicinity of the domain Ω . Linear continuous operator E acts on a function $w(x)$ from $H^7(\Omega)$ to the space

$$\tilde{H}^7(\Omega_\delta) = \left\{ w^e(x) \in H^7(\Omega_\delta) \mid w^e(x) = 0, \quad x \in \Omega_\delta \setminus \Omega_{\frac{\delta}{2}} \right\},$$

such so $w(x) = w^e(x)$, if $x \in \Omega$. Let us consider the integral

$$I(t) = \int_{\Omega_\delta} \left(\sum_{0 < |\alpha| \leq 7} |D_x^\alpha w^e(t, x)|^2 \right) dx,$$

where t is fixed. The sufficient smallness $I(t)$ is derived from the construction of E , in case if the truncated norm in $H^7(\Omega)$ is sufficiently small. But in virtue of homogenous boundary conditions in $\tilde{H}^7(\Omega_\delta)$ it follows that integral $I(t)$ is the square of the norm in $\tilde{H}^7(\Omega_\delta)$, and hence Sobolev embedding theorem can be applied.

As for $w_t(t, x)$ the situation is more trivial because from (10) we obtain

$$\|w(t, \cdot)\|_{H^6(\Omega)} \leq \frac{C_2(\varphi, \psi)}{1+t}, \quad t \geq 0.$$

We solve the problem (1), (2), (7) with given initial conditions, then this solution is substituted to the *only* right part of the equality (7), and we obtain the boundary condition (3) for the initial-boundary value problem (1)—(3). In other words, we make the control function of the problem (1)—(3) be equal to

$$u(t, x) = -k \frac{\partial w_0(t, x)}{\partial t},$$

where w_0 is a solution to the problem (1), (2), (7).

Therefore it is proved (here we use Sobolev embedding theorem) that controlling for a long time, we can make the values

$$\|w(T_1, \cdot)\|_{C^5(\bar{\Omega})}, \quad \|w_t(T_1, \cdot)\|_{C^4(\bar{\Omega})}$$

arbitrary small at some time $t = T_1$.

Now let us show, that the boundary control function $u(t, x)$ can be sufficiently small, i.e. we may satisfy the restriction (4). Note that in virtue of (9)

$$\max_{t \in [0, +\infty)} E(t) = E(0) = \int_{\Omega} (\varphi_{x_1}^2(x) + \varphi_{x_2}^2(x) + \psi^2(x)) dx.$$

Therefore we obtain

$$\int_{\Omega} w_t^2(t, x) dx \leq \int_{\Omega} (\varphi_{x_1}^2(x) + \varphi_{x_2}^2(x) + \psi^2(x)) dx, \quad t \geq 0.$$

In the last estimate of the integral, which is in the right side, depends on only initial data and does not depend on the friction coefficient. Hence differentiating the equation (1), initial condition (2) and boundary conditions (3) with respect to x_1 , x_2 , and also using embedding theorems, we obtain the restriction of absolute value of $w_t(t, x)$ on the closure of cylinder Q_{T_1} by constant, which depends on only initial data. Choosing the coefficient k is small enough, we have the condition (4) done.

4 The second step of the control

Now we have a task to drive the system to rest. A pair of functions

$$(w|_{t=0} = w(T_1, x), w_t|_{t=0} = w_t(T_1, x))$$

is considered to be new initial data for the problem (1)—(3). Remind that according to the fact proven above these initial conditions are sufficiently small in the norm of the space $C^{5,4}(\overline{\Omega})$.

Again let us consider the domain Ω_δ , which is δ -vicinity of the domain Ω . Also let take an arbitrary pair $(w_0(x), w_1(x))$ from the space $C^{5,4}(\overline{\Omega})$. Extend functions $w_0(x)$ and $w_1(x)$ on the whole plane R^2 by means of the bounded linear operator E_1 so that supports of extended functions are in $\overline{\Omega}_\delta$, $i = 0, 1$. The construction of the operator of E_1 is identical to the construction of the operator E .

Extended in this way functions are denoted (as we did above) as $w_0^e(x)$ and $w_1^e(x)$, according to D. L. Russell.

Let us consider the Cauchy problem for the equation of membrane's oscillations on a plane R^2 :

$$w_{tt}(t, x) - \Delta w(t, x) = 0, \quad (t, x) \in Q = (0, +\infty) \times R^2, \quad (12)$$

$$w|_{t=0} = w_0^e(x), \quad w_t|_{t=0} = w_1^e(x), \quad x \in R^2. \quad (13)$$

It is known that the solution to the problem (9), (10) has the form (Poisson's formula):

$$w(t, x) = \frac{\partial}{\partial t} \left(\frac{1}{2\pi} \int_{|y-x|<t} \frac{w_0^e(y)dy}{\sqrt{t^2 - |y-x|^2}} \right) + \frac{1}{2\pi} \int_{|y-x|<t} \frac{w_1^e(y)dy}{\sqrt{t^2 - |y-x|^2}}. \quad (14)$$

We use the formula (14) for estimating the absolute value of the solution $w(t, x)$ uniformly by the initial data. The absolute value of $w(t, x)$ is estimated in case $x \in \overline{\Omega}_\delta$. Compute the first derivative with respect to t in the right part of (14):

$$w(t, x) = \frac{1}{2\pi t} \int_{|y-x|<t} \frac{w_0^e(y) + (y-x) \cdot \nabla w_0^e(y)}{\sqrt{t^2 - |y-x|^2}} dy + \frac{1}{2\pi} \int_{|y-x|<t} \frac{w_1^e(y)dy}{\sqrt{t^2 - |y-x|^2}}. \quad (15)$$

As initial data $(w_0^e(x), w_1^e(x))$ have a compact support then there is large enough time $t^* > 0$ such that for any $t > t^*$ and for any $x \in \overline{\Omega}_\delta$ we obtain

$$w(t, x) = \frac{1}{2\pi t} \int_{\Omega_\delta} \frac{w_0^e(y) + (y-x) \cdot \nabla w_0^e(y)}{\sqrt{t^2 - |y-x|^2}} dy + \frac{1}{2\pi} \int_{\Omega_\delta} \frac{w_1^e(y)dy}{\sqrt{t^2 - |y-x|^2}}. \quad (16)$$

Note that we choose t such as $t^2 - |y-x|^2 \geq \alpha > 0$ for any $x, y \in \overline{\Omega}_\delta$.

The following rough evaluation follows from the explicit form of (16):

$$\|w(t, \cdot)\|_{C^5(\overline{\Omega}_\delta)} \leq \frac{C_1}{t} \|w_0^e\|_{C^5(R^2)} + \frac{C_2}{t} \|w_1^e\|_{C^4(R^2)}. \quad (17)$$

Differentiating $w(t, x)$ with respect to t , we obtain the rough estimate in the space of pair of functions $C^{5,4}(\overline{\Omega}_\delta) = C^5(\overline{\Omega}_\delta) \times C^4(\overline{\Omega}_\delta)$

$$\|(w(t, \cdot), w_t(t, \cdot))\|_{C^{5,4}(\overline{\Omega}_\delta)} \leq \frac{M}{t} \|(w_0^e, w_1^e)\|_{C^{5,4}(R^2)}, \quad t > 0, \quad (18)$$

where a number M does not depends on initial data.

Further we use the method, described in [2] and applied to problems of the boundary controllability for a wave equation.

Let us consider some initial conditions $w_0(x)$ and $w_1(x)$, $x \in \Omega$. We extend them to R^2 by means of a linear bounded operator E_1 . Then we obtain $w_0^e = E_1 w_0$, $w_1^e = E_1 w_1$. And the Cauchy problem (9),

(10) is arisen. Let $w^s(t, x)$ be the solution to this Cauchy problem. Now consider any **large enough** time $t = T_2$. We get $(w^s(T_2, x), w_t^s(T_2, x)) \in C^{5,4}(R^2)$. The restriction of the function $w^s(T_2, x)$ and its derivative on the domain Ω should be considered. It is obvious that in virtue of (18) the following estimate is correct for $t = T_2$

$$\|(w^s(T_2, \cdot), w_t^s(T_2, \cdot))\|_{C^{5,4}(\overline{\Omega})} \leq \frac{M}{T_2} \|(w_0^e, w_1^e)\|_{C^{5,4}(R^2)}. \quad (19)$$

Let by definition $w_0^{s,e}(T_2, x) = E_1(w^s(T_2, x)|_\Omega)$ and $w_1^{s,e}(T_2, x) = E_1(w_t^s(T_2, x)|_\Omega)$. Now let us have a look at the inverse Cauchy problem with initial conditions

$$w(t, x)|_{t=T_2} = -w_0^{s,e}(T_2, x) \quad w_t(t, x)|_{t=T_2} = -w_1^{s,e}(T_2, x). \quad (20)$$

Let $w^i(t, x)$ be the solution to the inverse Cauchy problem with conditions (20). In virtue of invertibility of the equation (1) with respect to t the following estimate takes place:

$$\|(w^i(0, \cdot), w_t^i(0, \cdot))\|_{C^{5,4}(\overline{\Omega})} \leq \frac{M}{T_2} \|(w_0^{s,e}(T_2, x), w_1^{s,e}(T_2, x))\|_{C^{5,4}(R^2)}. \quad (21)$$

Obviously the solution of the Cauchy problem with initial conditions such as

$$w|_{t=0} = w_0^e(x) + w^i(0, x), \quad w_t|_{t=0} = w_1^e(x) + w_t^i(0, x), \quad x \in R^2, \quad (22)$$

identically equals zero in Ω as well as its first derivative with respect to t at the time $t = T_2$. Now let us consider the restriction of the right parts of (22) in the domain Ω . We regard initial conditions (the restriction of right parts of (22) in the domain Ω) in the problem of boundary controllability:

$$w|_{t=0} = w_0(x) + w^{i,r}(0, x), \quad w_t|_{t=0} = w_1(x) + w_t^{i,r}(0, x), \quad x \in \Omega. \quad (23)$$

Note that it is the value of the corresponding solution to the Cauchy problem in R^2 with initial conditions (22) to determine the required control function on the boundary of Ω .

A pair $(w^{i,r}(0, x), w_t^{i,r}(0, x))$ is derived from pair $(w_0(x), w_1(x))$ by means of applying a linear continuous operator, let us denote it as L , with the norm less than 1 (consequence from estimates (19) and (21)). Obviously the sums in right parts (23) generate all elements of the space $C^{5,4}(\overline{\Omega})$. Indeed, (23) can be written as:

$$(I + L)(w_0(x), w_1(x)) = (w|_{t=0}, w_t|_{t=0}), \quad (24)$$

where I is the identical operator. Hence, as $\|L\| < 1$, so the operator $I + L$, which acts from $C^{5,4}(\overline{\Omega})$ to itself, is invertible.

Now let us represent the control function (second step) in the following form:

$$u(t, x) = \frac{\partial}{\partial \nu} K_+^t \left[\left(I + (-K_-^{T_2}) E_1 R K_+^{T_2} \right) E_1 \left(I + R (-K_-^{T_2}) E_1 R K_+^{T_2} E_1 \right)^{-1} \{w|_{t=0}, w_t|_{t=0}\} \right], \quad x \in \Omega,$$

where R is a restriction from R^2 to Ω and $K_+^{T_2}$, $K_-^{T_2}$ are resolving operators of the Cauchy problem. We write minus before $K_-^{T_2}$ because of (20).

Thus we have proven that the system with smooth initial conditions can be driven to rest by means of extending them on the full plane. It is the method to extend determines a program of the boundary control. Let us show now that if initial conditions have small enough absolute values, we can drive the system to rest by means of a boundary control which has a small absolute value.

We regard that in the problem (1)–(3) the value of the solution $w(t, x)$ and the value of its derivative $w_t(t, x)$ at $t = T_1$ are small enough in norms of spaces $C^5(\overline{\Omega})$ and $C^4(\overline{\Omega})$ respectively.

Let $(w|_{t=0}, w_t|_{t=0})$ be rewritten according to the formula (23). As continuous operator $I + L$ invertible, so according to Banach's theorem an invertible operator is continuous too. Hence choosing $(w|_{t=0}, w_t|_{t=0})$ sufficiently small, we can make $(w_0(x), w_1(x))$ be sufficiently small as well. Now let consider the sums (23), which determine data $(w|_{t=0}, w_t|_{t=0})$. Extending these sums on the whole plane by the method above, we obtain initial data (22). So we can write down the solution to the Cauchy problem with initial conditions (22):

$$w(t, x) = \frac{\partial}{\partial t} \left(\frac{1}{2\pi} \int_{|y-x|<t} \frac{(w_0^e(y) + w^i(0, y))dy}{\sqrt{t^2 - |y-x|^2}} \right) + \frac{1}{2\pi} \int_{|y-x|<t} \frac{(w_1^e(y) + w_t^i(0, y))dy}{\sqrt{t^2 - |y-x|^2}}. \quad (25)$$

Let us rewrite the last formula by solving the inverse Cauchy problem:

$$\begin{aligned} w(t, x) &= \frac{\partial}{\partial t} \left(\frac{1}{2\pi} \int_{|y-x|<t} \frac{w_0^e(y)dy}{\sqrt{t^2 - |y-x|^2}} \right) + \frac{1}{2\pi} \int_{|y-x|<t} \frac{w_1^e(y)dy}{\sqrt{t^2 - |y-x|^2}} - \\ &\frac{\partial}{\partial t} \left(\frac{1}{2\pi} \int_{|y-x|<t-T_2} \frac{w_0^{s,e}(T_2, y)dy}{\sqrt{(t-T_2)^2 - |y-x|^2}} \right) - \frac{1}{2\pi} \int_{|y-x|<t-T_2} \frac{w_1^{s,e}(T_2, y)dy}{\sqrt{(t-T_2)^2 - |y-x|^2}}. \end{aligned} \quad (26)$$

In order to solve the Cauchy problem with initial conditions $w_0^e(x)$ and $w_1^e(x)$ the law of conservation of energy is used:

$$\begin{aligned} \int_{R^2} \left\{ (w_{x_1}^s(t, x))^2 + (w_{x_2}^s(t, x))^2 + (w_t^s(t, x))^2 \right\} dx = \\ \int_{R^2} \left\{ \left(\frac{\partial w_0^e(x)}{\partial x_1} \right)^2 + \left(\frac{\partial w_0^e(x)}{\partial x_2} \right)^2 + (w_1^e(x))^2 \right\} dx. \end{aligned} \quad (27)$$

Remind that supports of functions $w_0^e(x)$ and $w_1^e(x)$ are in $\overline{\Omega}_\delta$, and supports of their derivatives with respect to all variables including the third order are located in $\overline{\Omega}_\delta$ too. Note that $(w^s(t, x), w_t^s(t, x))$ is an element of $C^{4,3}(R^2)$ for any t .

Now differentiating the equation (12) and initial conditions (13) with respect to variables x_1, x_2 , we obtain the estimate

$$\|w^s(t, \cdot)\|'_{H^3(\Omega)} \leq \|w_0^e\|_{H^3(\Omega_\delta)} + \|w_1^e\|_{H^3(\Omega_\delta)},$$

where $\|\cdot\|'_{H^3(\Omega)}$ is a seminorm (term

$$\int_{\Omega} (w^s(t, x))^2 dx$$

is absent). Let us now extend the function $w^s(t, \cdot)$ from Ω to Ω_δ as we did it earlier. The linear bounded operator of extension we denote E_2 . As

$$\|w^{s,e}(t, \cdot)\|'_{H^3(\Omega_\delta)} \leq C_{E_2} \|w^s(t, \cdot)\|'_{H^3(\Omega)}$$

then we obtain

$$\|w^{s,e}(t, \cdot)\|'_{H^3(\Omega_\delta)} \leq C_{E_2} \|w_0^e\|_{H^3(\Omega_\delta)} + C_{E_2} \|w_1^e\|_{H^3(\Omega_\delta)}.$$

Note that C_{E_2} does not depend on t . Using the boundary conditions and Friedrichs' inequality we obtain

$$\|w^{s,e}(t, \cdot)\|_{H^3(\Omega_\delta)} \leq C_{E_2} C_F \|w_0^e\|_{H^3(\Omega_\delta)} + C_{E_2} C_F \|w_1^e\|_{H^3(\Omega_\delta)}.$$

Hence

$$\|w^s(t, \cdot)\|_{H^3(\Omega)} \leq C_{E_2} C_F \|w_0^e\|_{H^3(\Omega_\delta)} + C_{E_2} C_F \|w_1^e\|_{H^3(\Omega_\delta)}.$$

Taking into account the last estimate and the Sobolev embedding theorem we get

$$\|w^s(t, \cdot)\|_{C^1(\overline{\Omega})} \leq C_S \|w_0^e\|_{H^3(\Omega_\delta)} + C_S \|w_1^e\|_{H^3(\Omega_\delta)}. \quad (28)$$

Summing up it is proved that the solution $w^s(t, x)$ can be made sufficiently small in the norm $C^1(\overline{\Omega})$ for any $t \in [0, T_2]$. The same argument may be applied to the solution of the inverse Cauchy problem with initial conditions $-w_0^{s,e}(T_2, x)$ and $-w_1^{s,e}(T_2, x)$ (see (26)). In this case it is important that functions $w_0^{s,e}(T_2, x)$ and $w_1^{s,e}(T_2, x)$ in virtue of inequality (18) are "small" in $C^{5,4}$, if $w_0^e(x)$ and $w_1^e(x)$ are "small". Hence the restriction of normal derivative of the solution to the Cauchy problem (9), (10) on the boundary of Ω (Neumann condition of the problem of controllability) is less than given ε with respect to absolute value. The last means that the required restriction (4) on the control function $u(t, x)$ is satisfied.

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